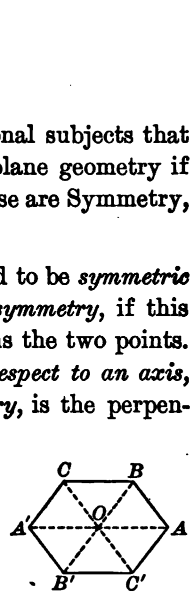
APPENDIX

405. Subjects Treated. Of the many additional subjects that may occupy the attention of the student of plane geometry if time permits, two are of special interest. These are Symmetry, and Maxima and Minima.

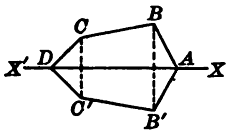
406. Symmetric Points. Two points are said to be *symmetric with respect to a point*, called the *center of symmetry*, if this third point bisects the straight line which joins the two points.

Two points are said to be *symmetric with respect to an axis*, if a straight line, called the *axis of symmetry*, is the perpendicular bisector of the line joining them.

407. Symmetric Figure. A figure is said to be *symmetric with respect to a point*, if the point bisects every straight line drawn through it and terminated by the boundary of the figure.

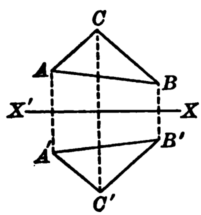


A figure is said to be *symmetric with respect to an axis*, if the axis bisects every perpendicular through it and terminated by the boundary of the figure.



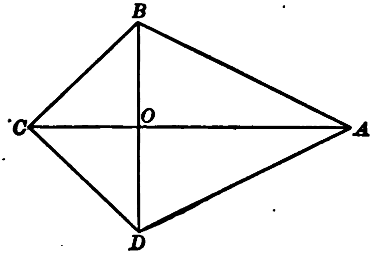
Evidently this will be the case if one part coincides with another part when folded over the axis.

408. Two Symmetric Figures. Two figures are said to be *symmetric with respect to a point* or *symmetric with respect to an axis*, if every point of each has a corresponding symmetric point in the other.



Proposition I. Theorem

409. *A quadrilateral that has two adjacent sides equal, and the other two sides equal, is symmetric with respect to the diagonal joining the vertices of the angles formed by the equal sides; and the diagonals are perpendicular to each other.*



Given the quadrilateral *ABCD*, having *AB* equal to *AD*, and *CB* equal to *CD*, and having the diagonals *AC* and *BD*.

To prove that the diagonal AC is an axis of symmetry, and that AC is ⊥ to BD.

Proof. In the $\triangle$ ABC and ADC,

	$AB = AD$, and $CB = CD$,	Given
and	$AC = AC$.	Iden.
	$\therefore \triangle ABC$ is congruent to $\triangle ADC$.	§ 80
	$\therefore \angle BAC = \angle CAD$, and $\angle ACB = \angle DCA$.	§ 67

Hence, if $\triangle ABC$ is turned on AC as an axis until it falls on $\triangle ADC$, AB will fall on AD, CB on CD, and OB on OD.

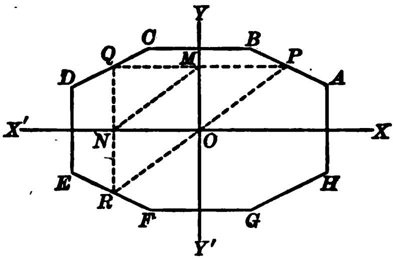
$\therefore$ the $\triangle ABC$ will coincide with the $\triangle ADC$.

$\therefore AC$ will bisect every perpendicular drawn through it and terminated by the boundary of the figure.

$\therefore AC$ is an axis of symmetry.	§ 407
$\therefore AC$ is $\perp$ to BD, by § 406.	Q. E. D.

PROPOSITION II. THEOREM

410. *If a figure is symmetric with respect to two axes perpendicular to each other, it is symmetric with respect to their intersection as a center.*



Given the figure *ABCDEFGH*, symmetric with respect to the two perpendicular axes *XX'*, *YY'*, which intersect at *O*.

To prove that O is the center of symmetry of the figure.

Proof. Let P be any point in the perimeter.

Draw $PMQ \perp$ to YY', and $QNR \perp$ to XX'. § 227

Then PQ is $\parallel$ to XX', and QR is $\parallel$ to YY'. § 95

Draw PO, OR, and MN.

Then $QN = NR$. § 407

(*The figure is given as symmetric with respect to XX'.*)

But $QN = MO$. § 127

$\therefore NR = MO$. Ax. 8

$\therefore RO$ is equal and parallel to NM. § 130

In like manner, OP is equal and parallel to NM.

$\therefore ROP$ is a straight line. § 94

$\therefore O$ bisects PR, any straight line, and hence bisects every straight line drawn through O and terminated by the perimeter.

$\therefore O$ is the center of symmetry of the figure, by § 407. Q.E.D.

EXERCISE 72

1. Draw a figure showing the number of axes of symmetry possessed by a square.

2. Draw a figure showing the number of axes of symmetry possessed by a regular hexagon.

3. Draw a figure showing six of the unlimited number of axes of symmetry of a circle, and showing the center of symmetry.

4. Show by drawings that two congruent triangles may be placed in a position of symmetry with respect to an axis. In one of the drawings let a common side be the axis.

5. Show by a drawing that two congruent triangles may be placed in a position of symmetry with respect to a center.

6. Two figures symmetric with respect to an axis are congruent.

7. Two figures symmetric with respect to a center are congruent.

8. Make a list of quadrilaterals that are symmetric with respect to an axis.

9. Make a list of quadrilaterals that are symmetric with respect to a center.

10. What kinds of regular polygons are symmetric with respect both to a center and to an axis? Prove this for the hexagon.

11. A circle is symmetric with respect to its center as a center of symmetry, and is also symmetric with respect to any diameter as an axis.

12. An isosceles triangle is symmetric with respect to an axis, and therefore the angles opposite the equal sides are equal.

13. Two tangents drawn to a circle from the same point are symmetric with respect to an axis.

14. The four common tangents to two given circles form, together with the circles, a figure symmetric with respect to the line of centers as an axis.

411. Maxima and Minima. Among geometric magnitudes that satisfy given conditions, the *greatest* is called the *maximum*, and the *smallest* is called the *minimum*.

The plural of maximum is *maxima*, and the plural of minimum is *minima*.

Among geometric magnitudes that satisfy given conditions, there may be several equal magnitudes that are greater than any others. In this case all are called maxima.

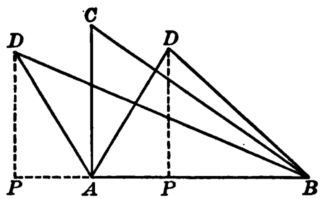
Similarly there may be several minima magnitudes of a given kind.

412. Isoperimetric Polygons. Polygons which have equal perimeters are called *isoperimetric polygons*.

If the circumference of a circle equals the perimeter of a polygon, the circle and the polygon are said to be isoperimetric, and similarly for all other closed figures in a plane.

Proposition III. Theorem

413. *Of all triangles having two given sides, that in which these sides include a right angle is the maximum.*



Given the triangles *ABC* and *ABD*, with *AB* and *CA* equal to *AB* and *DA* respectively, and with angle *BAC* a right angle.

To prove that $\triangle ABC > \triangle ABD$.

Proof.	From D draw the altitude DP.	§ 227
Then	$DA > DP$.	§ 86
But	$DA = CA$.	Given
	$\therefore CA > DP$.	Ax. 9
	$\therefore \triangle ABC > \triangle ABD$, by § 327.	Q.E.D.

Proposition IV. Theorem

414. *Of all isoperimetric triangles having the same base the isosceles triangle is the maximum.*

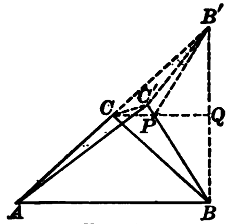


Fig. 1

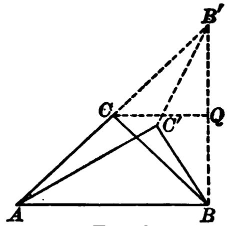


Fig. 2

Given the triangles ABC and ABC' having equal perimeters, and having AC equal to BC, and AC' not equal to BC'.

To prove that $\triangle ABC > \triangle ABC'$.

Proof. Produce AC to B', making $CB' = AC$.

Draw BB' and $C'B'$, and draw $CQ \parallel$ to AB.

Then since $AC = CB'$, $\therefore BQ = QB'$. § 135

And since $CA = CB = CB'$, $\therefore \angle B'BA$ is a rt. $\angle$. § 215

$\therefore CQ$ is $\perp$ to BB'. § 97

C' cannot lie on AB', for if it could, then $CC' + C'B$ would equal CB, which is impossible. Post. 1

Then since $AC + CB' < AC' + C'B'$, § 112

$\therefore AC + CB < AC' + C'B'$. Ax. 9

$\therefore AC' + C'B < AC' + C'B'$. Ax. 9

$\therefore C'B < C'B'$. Ax. 6

$\therefore C'$ cannot lie on CQ, for then $C'B$ would equal $C'B'$. § 150

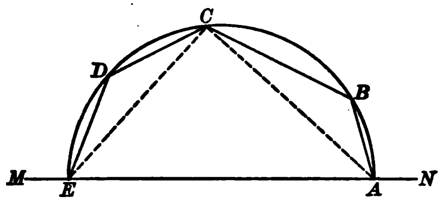
C' cannot lie above CQ (Fig. 1), for $C'B'$, which $< C'P + PB'$, would be less than $C'B$, which equals $C'P + PB'$.

$\therefore C'$ must lie below CQ, as in Fig. 2.

$\therefore \triangle ABC > \triangle ABC'$, by § 327. Q.E.D.

Proposition V. Theorem

415. *Of all polygons with sides all given but one, the maximum can be inscribed in the semicircle which has the undetermined side for its diameter.*



Given *ABCDE*, the maximum of polygons with sides *AB*, *BC*, *CD*, *DE*, having the vertices *A* and *E* on the line *MN*.

To prove that ABCDE can be inscribed in the semicircle having EA for its diameter.

Proof. From any vertex, as C, draw CA and CE.

The $\triangle ACE$ must be the maximum of all $\triangle$s having the sides CA and CE, and the third side on MN; otherwise, by increasing or diminishing the $\angle ECA$, keeping the lengths of the sides CA and CE unchanged, but sliding the extremities A and E along the line MN, we could increase the $\triangle ACE$, while the rest of the polygon would remain unchanged; and therefore we could increase the polygon. But this is contrary to the hypothesis that the polygon is the maximum polygon.

Hence the $\triangle ACE$ is the maximum of triangles that have the sides CA and CE.

Therefore the $\angle ACE$ is a right angle. § 413

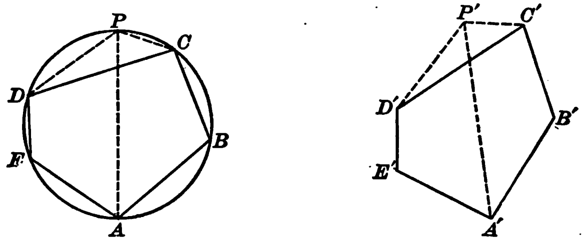
Therefore C lies on the semicircle having EA for its diameter. § 215

Hence every vertex lies on this semicircle.

That is, the maximum polygon can be inscribed in the semicircle having the undetermined side for its diameter. Q. E. D.

PROPOSITION VI. THEOREM

416. *Of all polygons with given sides, one that can be inscribed in a circle is the maximum.*



Given the polygon *ABCDE* inscribed in a circle, and the polygon *A′B′C′D′E′* which has its sides equal respectively to the sides of *ABCDE*, but which cannot be inscribed in a circle.

To prove that $ABCDE > A'B'C'D'E'$.

Proof. Draw the diameter AP, and draw CP and PD.

Upon $C'D'$ as a base, construct the $\triangle C'P'D'$ congruent to the $\triangle CPD$, and draw $A'P'$.

Since, by hypothesis, a $\odot$ cannot pass through all the vertices of $A'B'C'P'D'E'$, one or both of the parts $A'P'D'E'$, $A'B'C'P'$ cannot be inscribed in a semicircle.

Neither $A'P'D'E'$ or $A'B'C'P'$ can be greater than its corresponding part. § 415

(*Of all polygons with sides all given but one, the maximum can be inscribed in the semicircle which has the undetermined side for its diameter.*)

Therefore one of the parts $A'P'D'E'$, $A'B'C'P'$ must be less than, and the other cannot be greater than, the corresponding part of $ABCPDE$.

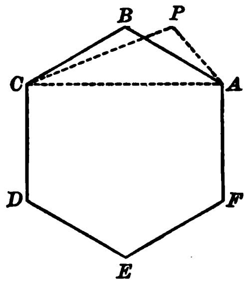
$$\therefore ABCPDE > A'B'C'P'D'E'.$$

Take from the two figures the congruent $\triangle$ CPD and $C'P'D'$.

Then $ABCDE > A'B'C'D'E'$, by Ax. 6. Q.E.D.

Proposition VII. Theorem

417. *Of isoperimetric polygons of a given number of sides, the maximum is equilateral.*



Given the polygon *ABCDEF*, the maximum of isoperimetric polygons of *n* sides.

To prove that the polygon ABCDEF is equilateral.

Proof. Draw AC.

The $\triangle ABC$ must be the maximum of all the $\triangle$s which are formed upon AC with a perimeter equal to that of $\triangle ABC$.

Otherwise a greater $\triangle APC$ could be substituted for $\triangle ABC$, without changing the perimeter of the polygon.

But this is inconsistent with the fact that the polygon $ABCDEF$ is given as the maximum polygon.

$\therefore$ the $\triangle ABC$ is isosceles. § 414

$\therefore AB = BC.$

Similarly $BC = CD$, $CD = DE$, and so on.

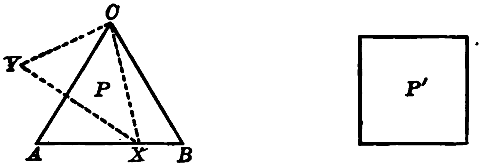
$\therefore$ the polygon $ABCDEF$ is equilateral. Q.E.D.

418. Corollary. *The maximum of isoperimetric polygons of a given number of sides is a regular polygon.*

For the maximum polygon is equilateral (§ 417), and can be inscribed in a circle (§ 416). Therefore the maximum polygon is regular (§ 365).

Proposition VIII. Theorem

419. *Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*



Given the regular polygon P of three sides, and the isoperimetric regular polygon P' of four sides.

To prove that $P' > P$.

Proof. Draw CX from C to any point X in AB.

Invert the $\triangle AXC$ and place it in the position XCY, letting X fall at C, C at X, and A at Y.

The polygon $XBCY$ is an irregular polygon of four sides, which by construction has the same perimeter as P' and the same area as P

Then the regular polygon P' of four sides is greater than the isoperimetric irregular polygon $XBCY$ of four sides. § 418

That is, a regular polygon of four sides is greater than the isoperimetric regular polygon of three sides.

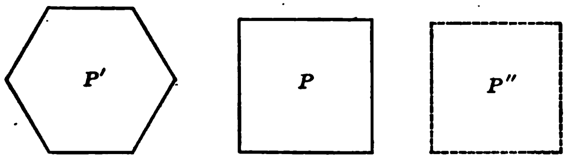
In like manner, it may be shown that P' is less than the isoperimetric regular polygon of five sides, and so on. Q.E.D.

Discussion. We may illustrate this by the case of an equilateral triangle and a square, each with the perimeter p. In the triangle the base is $\frac{1}{3}p$, the altitude $\frac{1}{6}p\sqrt{3}$, and the area $\frac{1}{36}p^2\sqrt{3}$, or about $0.048\,p^2$. In the square the base and altitude are each $\frac{1}{4}p$, and the area is $\frac{1}{16}p^2$, or $0.0625\,p^2$. The area of the polygon is therefore increasing as we increase the number of sides.

Since the limit approached by the perimeters is a circle, we may infer that *of all isoperimetric plane figures the circle has the greatest area.*

Proposition IX. Theorem

420. *Of regular polygons having a given area, that which has the greatest number of sides has the minimum perimeter.*



Given the regular polygons P and P' having the same area, P' having the greater number of sides.

To prove that the perimeter of $P >$ the perimeter of P'.

Proof. Construct the regular polygon P'' having the same perimeter as P', and the same number of sides as P.

Denote a side of P by s, and a side of P'' by s''.

Then	$P' > P''$.	§ 419
But	$P = P'$.	Given
	$\therefore P > P''$.	Ax. 9
But	$P : P'' = s^2 : s''^2$.	§ 374
	$\therefore s^2 > s''^2$.	
	$\therefore s > s''$.	Ax. 6

$\therefore$ the perimeter of $P >$ the perimeter of P''. Ax. 6

But the perimeter of $P' =$ the perimeter of P''. Const.

$\therefore$ the perimeter of $P >$ the perimeter of P', by Ax. 9. Q.E.D.

Discussion. We may illustrate this, as on page 270, by the case of an equilateral triangle and a square, each with area a^2. The side of the square is a, and the perimeter $4a$. The area of the equilateral triangle is $\frac{1}{4}s^2\sqrt{3}$. Therefore $\frac{1}{4}s^2\sqrt{3} = a^2$, or $\frac{1}{2}s\sqrt[4]{3} = a$. Now $\sqrt[4]{3} = \sqrt{\sqrt{3}}$; hence we have $\sqrt{3} = 1.73+$, and $\sqrt{\sqrt{3}} = \sqrt{1.73} = 1.3+$. Hence $\frac{1}{2}s \times 1.3 = a$, and $s = 1.5a$, and the perimeter of the triangle is $4.5a$. Therefore the perimeter of the square is less than that of the triangle.

EXERCISE 73

Maxima and Minima

1. Of all equivalent parallelograms that have equal bases, the rectangle has the minimum perimeter.

2. Of all equivalent rectangles, the square has the minimum perimeter.

3. Of all triangles that have the same base and the same altitude, the isosceles has the minimum perimeter.

4. Of all triangles that can be inscribed in a given circle, the equilateral is the maximum and has the maximum perimeter.

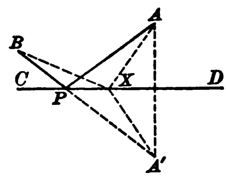
5. To inscribe in a semicircle the maximum rectangle.

6. Find the area of the maximum triangle inscribed in a semicircle whose radius is 3 in.

7. Of all polygons of a given number of sides that can be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

8. Of all polygons of a given number of sides that can be circumscribed about a given circle, that which is regular has the minimum area and the minimum perimeter.

9. In a given line required to find a point such that the sum of its distances from two given points on the same side of the line shall be the minimum.



How does $AP + PB$ compare with $A'B$? and this with $A'X + XB$? and this with $AX + XB$? This is the problem of a ray of light from A to the mirror CD, and reflected to B.

10. To divide a given line into two segments such that the sum of the squares on these segments shall be the minimum.

11. To divide a given line into two segments such that their product shall be the maximum.